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Backward stochastic differential equations with reflection and weak assumptions on the coefficients[☆]

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Abstract

In this paper, we study reflected BSDE's with one continuous barrier, under monotonicity and general increasing conditions in y and non-Lipschitz conditions in z. We prove the existence and uniqueness of a solution by an approximation method.

Keywords: Reflected backward stochastic differential equation; Monotonicity; Non-Lipschitz condition; Quadratic increasing; Linear increasing

1. Introduction

Nonlinear backward stochastic differential equations (BSDE's for short) were introduced by Pardoux and Peng in 1990 [11]. They proved that there exists a unique solution to this equation if the terminal condition ξ and the coefficient f satisfy smooth square integrability assumptions and if $f(t,\omega,y,z)$ is Lipschitz in (y,z) uniformly in (t,ω). Later, many assumptions were considered to relax the Lipschitz condition on f. Pardoux (1999 [10]) and Briand et al. (2003 [1]) considered the case of f Lipschitz in z but only with some monotonicity and general increasing in y, i.e. for some continuous increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, real number $\mu > 0$:

$$|f(t,y,0)| \le |f(t,0,0)| + \varphi(|y|), \quad \forall (t,y) \in [0,T] \times \mathbb{R}, \text{ a.s.}; \tag{1}$$

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$$(y-y')(f(t,y,z)-f(t,y',z)) \le \mu(y-y')^2, \quad \forall (t,z) \in [0,T] \times \mathbb{R}^d,\ y, y' \in \mathbb{R}, \text{a.s.}$$

The case f quadratic in z and linear in y, ξ bounded, has been studied by Kobylanski [5]. In [8], Lepeltier and San Martín generalized to a superlinear case in y. More recently [2], Braind et al. considered the case when f satisfies only monotonicity, continuity and generalized increasing in y, and quadratic or linear increasing in z, i.e.

$$\begin{aligned} &(y-y')(f(t,y,z)-f(t,y',z)) \le \mu(y-y')^2, \quad \forall (t,z) \in [0,T] \times \mathbb{R}^d,\ y, y' \in \mathbb{R}, \text{ a.s.} \\ &|f(t,y,z)| \le \varphi(|y|) + A\,|z|^2, \quad \forall (t,y) \in [0,T] \times \mathbb{R}, \text{ a.s.}; \end{aligned} \tag{2}$$

or

$$|f(t,y,z)| \le g_t + \varphi(|y|) + A\,|z|, \quad \forall (t,y) \in [0,T] \times \mathbb{R}, \text{ a.s.} \tag{3}$$

El Karoui, Kapoudjian, Pardoux, Peng and Quenez introduced in 1997 the notion of reflected BSDE (RBSDE for short) on one lower barrier [4]: the solution is forced to remain above a continuous process, which is considered as the lower barrier. More precisely, a solution for such an equation associated with a coefficient f, a terminal value ξ, a continuous barrier L, is a triple $(Y_t, Z_t, K_t)_{0 \le t \le T}$ of adapted processes valued on $\mathbb{R}^{1+d+1}$, which satisfies a square integrability condition,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s, \quad 0 \le t \le T, \text{a.s.},$$

and $Y_t \ge L_t$, $0 \le t \le T$, a.s. Furthermore, the process $(K_t)_{0 \le t \le T}$ is non-decreasing, continuous, and the role of K_t is to push upward the state process in a minimal way, to keep it above L. In this sense it satisfies $\int_0^T (Y_s - L_s)\mathrm{d}K_s = 0$. They proved existence and uniqueness of a solution when f is Lipschitz in (y, z) uniformly in (t, ω). Then Matoussi (1997 [9]) considered the case f continuous and at most linear growth in y, z and proved the existence of a maximal and a minimal solution.

In [6], Kobylanski, Lepeltier, Quenez and Torres proved the existence of a maximal and minimal bounded solution for the RBSDE when the coefficient $f(t, \omega, y, z)$ is superlinear increasing in y and quadratic in z, i.e. there exists a function l strictly positive such that

$$|f(t,y,z)| \le l(y) + A\,|z|^2, \quad \text{with } \int_0^\infty \frac{\mathrm{d}x}{l(x)} = +\infty.$$

In this case, ξ and L are required to be bounded, and L is a continuous process. Recently, in [7] Lepeltier, Matoussi and Xu considered the case when $f(t, \omega, y, z)$ satisfies (1) and is Lipschitz in z. They proved the existence and uniqueness of the solution by using an approximation procedure.

In this paper, we study the case when the coefficient f satisfies the conditions (2) or (3), and the lower barrier L is uniformly bounded. We prove the existence of a solution, following the methods in [2], and we give a necessary and sufficient condition for the case when $f(t, \omega, y, z) = |z|^2$.

The paper is organized as follows. In Section 2, we present the basic assumptions and recall the notion of RBSDE; then in Section 3, we prove the existence of a solution when f satisfies (2), ξ and L are bounded; in the following section, we consider the case when $f(t, \omega, y, z) = |z|^2$, and ξ is not necessarily bounded. Finally, in Section 5, we study the RBSDE with condition (3), and prove the existence of a solution. Finally, in the appendix, we generalize the comparison theorem proved in [6], and get some comparison theorems, which help us to pass to the limit in the approximations.

2. Notation

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $(B_t)_{0\le t\le T} = (B_t^1, B_t^2, \dots, B_t^d)'_{0\le t\le T}$ be a d-dimensional Brownian motion defined on a finite interval $[0, T]$, $0 < T < +\infty$. Let $\{\mathcal{F}_t; 0 \le t \le T\}$ be the standard filtration generated by the Brownian motion B, i.e. $\mathcal{F}_t$ is the completion of

$$\mathcal{F}_t = \sigma\{B_s; 0 \le s \le t\},$$

with respect to $(\mathcal{F}, P)$. We denote by $\mathcal{P}$ the σ-algebra of predictable sets on $[0, T] \times \Omega$.

We shall need the following spaces:

$$\mathbf{L}^2(\mathcal{F}_t) = \{\eta : \mathcal{F}_t\text{-measurable real-valued variable, s.t. } E(|\eta|^2) < +\infty\},$$

$$\mathbf{H}_n^2(0, T) = \left\{(\psi_t)_{0\le t\le T} : \text{predictable process valued in } \mathbb{R}^n\text{, s.t. } E\int_0^T |\psi(t)|^2 \mathrm{d}t < +\infty\right\},$$

$$\begin{aligned}\mathbf{S}^2(0, T) &= \{(\psi_t)_{0\le t\le T} : \text{progressively measurable, continuous, real-valued process,}\\ &\qquad \text{s.t. } E(\sup_{0\le t\le T} |\psi(t)|^2) < +\infty\},\end{aligned}$$

$$\begin{aligned}\mathbf{A}^2(0, T) &= \{(K_t)_{0\le t\le T} : \text{ adapted continuous increasing process,}\\ &\qquad \text{s.t. } K(0) = 0, E(K(T)^2) < +\infty\}.\end{aligned}$$

Now we introduce the definition of a solution for a RBSDE with terminal condition ξ, coefficient f and continuous reflecting lower barrier L (RBSDE(ξ, f, L) for short), which is the same as in El Karoui et al. [4].

Definition 2.1. We say that the triple $(Y_t, Z_t, K_t)_{0\le t\le T}$ of progressively measurable processes is solution of the RBSDE(ξ, f, L), if the following hold:

(i) $(Y_t)_{0\le t\le T} \in \mathbf{S}^2(0, T)$, $(Z_t)_{0\le t\le T} \in \mathbf{H}_d^2(0, T)$, and $(K_t)_{0\le t\le T} \in \mathbf{A}^2(0, T)$.
(ii) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s$, $0 \le t \le T$ a.s.
(iii) $Y_t \ge L_t$, $0 \le t \le T$.
(iv) $\int_0^T (Y_s - L_s)\mathrm{d}K_s = 0$, a.s.

3. The general case: f quadratic increasing

In this section, we work under the following assumptions:

Assumption 1. ξ is $\mathcal{F}_T$-adapted and bounded.

Assumption 2. $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is such that there exists some continuous increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, real numbers μ and $A > 0$ such that $\forall (t, y, y'z) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$;

(i) $f(\cdot, y, z)$ is progressively measurable;
(ii) $|f(t, y, z)| \le \varphi(|y|) + A|z|^2$;
(iii) $(y - y')(f(t, y, z) - f(t, y', z)) \le \mu(y - y')^2$;
(iv) $y \to f(t, y, z)$ is continuous, a.s.

Assumption 3. The barrier $(L_t)_{0\le t\le T}$ is a bounded continuous progressively measurable real-valued process, $b := \sup_{0\le t\le T} |L_t| < +\infty$, $L_T \le \xi$, a.s.

The main result in this section is the following:

Theorem 3.1. *Under Assumptions 1–3, the RBSDE(ξ, f, L) has a maximal bounded solution.*

Proof. First, notice that (Y, Z, K) is a solution of the RBSDE(ξ, f, L) if and only if (Y^b, Z^b, K^b) is a solution of the RBSDE(ξ^b, f^b, L^b), where

$$(Y^b, Z^b, K^b) = (Y - b, Z, K),$$

and

$$(\xi^b, f^b(t, y, z), L^b) = (\xi - b, f(s, y + b, z), L - b).$$

The triple (ξ^b, f^b, L^b) satisfies Assumptions 1 and 2 and $-2b \le L^b \le 0$. So in the following, we assume that the barrier L is a negative bounded process.

For $C > 0$, set $g^C : \mathbb{R} \to \mathbb{R}$ to be a continuous function such that $0 \le g^C(y) \le 1$, $\forall y \in \mathbb{R}$, and

$$\begin{aligned} g^C(y) &= 1, \quad \text{if } |y| \le C, \\ g^C(y) &= 0, \quad \text{if } |y| \ge 2C. \end{aligned} \tag{4}$$

Define $f^C(t, y, z) = g^C(y) f(t, y, z)$; then

$$|f^C(t, y, z)| \le 1_{[-2C,2C]}(y)(\varphi(|y|) + A|z|^2) \le \varphi(2C) + A|z|^2.$$

From Theorem 1 in [6], there exists a maximal solution (Y^C, Z^C, K^C) of RBSDE(ξ, f^C, L), i.e.

$$\begin{aligned} Y_t^C &= \xi + \int_t^T g^C(Y_s^C) f(s, Y_s^C, Z_s^C)\mathrm{d}s - \int_t^T Z_s^C \mathrm{d}B_s + K_T^C - K_t^C, \\ Y_t^C &\ge L_t, \quad \int_0^T (Y_t^C - L_t)\mathrm{d}K_t^C = 0. \end{aligned} \tag{5}$$

We choose $n \ge 2$ even, and $a \in \mathbb{R}$; applying Itô's formula to $\mathrm{e}^{at}(Y_t^C)^n$, with the same techniques as for Theorem 2.1 in [2], and the fact the L is a negative bounded process; then we get

$$|Y_t^C| \le (\mathrm{e}^{(\varphi(0)+\mu)T} \vee 1)(\|\xi\|_\infty + 1).$$

If C is chosen to satisfy $C \ge (\mathrm{e}^{(\varphi(0)+\mu)T} \vee 1)(\|\xi\|_\infty + 1)$, then we have $\left|Y_t^C\right| \le C$, which implies $g^C(Y_t^C) = 1$, for $0 \le t \le T$. So, (Y^C, Z^C, K^C) is the solution of the RBSDE(ξ, f, L). □

4. The case $f(t, y, z) = |z|^2$

In this section we consider the case $f(t, y, z) = |z|^2$, which corresponds to the RBSDE

$$\begin{aligned} Y_t &= \xi + \int_t^T |Z_s|^2 \mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s, \\ Y_t &\ge L_t, \quad \int_0^T (Y_t - L_t)\mathrm{d}K_t = 0. \end{aligned} \tag{6}$$

Then we have

Theorem 4.1. *If $E(\sup_{0\le t\le T} \mathrm{e}^{2L_t}) < +\infty$, the RBSDE$(\xi, f, L)$ (6) has a solution if and only if $E(\mathrm{e}^{2\xi}) < +\infty$.*

Proof. For the necessary part, let (Y, Z, K) be solution of the RBSDE (6). By Itô's formula, we get

$$\begin{aligned}\mathrm{e}^{2Y_t} &= \mathrm{e}^{2\xi} + 2\int_t^T \mathrm{e}^{2Y_s}\mathrm{d}K_s - 2\int_t^T \mathrm{e}^{Y_s} Z_s \mathrm{d}B_s \\ &= \mathrm{e}^{2Y_0} + 2\int_0^t \mathrm{e}^{2Y_s} Z_s \mathrm{d}B_s - 2\int_0^t \mathrm{e}^{2Y_s}\mathrm{d}K_s. \end{aligned}\tag{7}$$

For all n, let $\tau_n = \inf\{t : Y_t \ge n\} \wedge T$; then $M_{t\wedge\tau_n} = 2\int_0^{t\wedge\tau_n} \mathrm{e}^{2Y_s} Z_s \mathrm{d}B_s$ is a martingale, and we have

$$E[\mathrm{e}^{2Y_{\tau_n}}] = E\left[\mathrm{e}^{2Y_0} - 2\int_0^t \mathrm{e}^{2Y_s}\mathrm{d}K_s\right] \le E[\mathrm{e}^{2Y_0}],$$

in view of $2\int_0^t \mathrm{e}^{2Y_s}\mathrm{d}K_s \ge 0$. Finally, since $\tau_n \nearrow T$ when $n \to \infty$,

$$E[\varliminf_{n\to\infty} \mathrm{e}^{2Y_{\tau_n}}] = E[\mathrm{e}^{2\xi}] \le E[\mathrm{e}^{2Y_0}] < \infty$$

follows from Fatou's Lemma.

Conversely if $E(\mathrm{e}^{2\xi}) < +\infty$, set $\widetilde{L}_t = L_t 1_{\{t<T\}} + \xi 1_{\{t=T\}}$ and

$$N_t = S_t(\mathrm{e}^{2\widetilde{L}}) = \operatorname*{ess\ sup}_{\tau\in\mathcal{T}_{t,T}} E[\mathrm{e}^{2\widetilde{L}_\tau}|\mathcal{F}_t],$$

where $S_t(\eta)$ denotes the Snell envelope of η (see El Karoui [3]), $\mathcal{T}_{t,T}$ is the set of all stopping times valued in $[t, T]$. Since

$$E[\sup_{0\le t\le T} \mathrm{e}^{2\widetilde{L}_t}] \le E[\sup_{0\le t\le T} \mathrm{e}^{2L_t} + \mathrm{e}^{2\xi}] < +\infty,$$

using the results about the Snell envelope, we know that N is a supermartingale, which admits the following decomposition:

$$N_t = N_0 + \int_0^t \overline{Z}_s \mathrm{d}B_s - \overline{K}_t$$

for an increasing integrable process $\overline{K}$. Applying Itô's formula to $\log N_t$, we get

$$\frac{1}{2}\log N_t = \frac{1}{2}\log N_0 + \frac{1}{2}\int_0^t \frac{\overline{Z}_s}{N_s}\mathrm{d}B_s - \frac{1}{4}\int_0^t \left(\frac{\overline{Z}_s}{N_s}\right)^2 \mathrm{d}s - \frac{1}{2}\int_0^t \frac{1}{N_s}\mathrm{d}\overline{K}_s.$$

Set $Y_t = \frac{1}{2}\log N_t$, $Z_t = \frac{\overline{Z}_t}{2N_t}$, $K_t = \frac{1}{2}\int_0^t \frac{1}{N_s}\mathrm{d}\overline{K}_s$; then the triple satisfies

$$Y_t = \xi + \int_t^T Z_s^2 \mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s. \tag{8}$$

Thanks to the results about the Snell envelope, we know that $N_t \ge \mathrm{e}^{2\widetilde{L}_t}$ and $\int_0^T (N_t - \mathrm{e}^{2\widetilde{L}_t})\mathrm{d}\overline{K}_t = 0$. The first inequality implies

$$Y_t \ge \widetilde{L}_t \ge L_t.$$

From another part, $N_t > 0$, $0 \le t \le T$, so K is increasing. If we consider the stopping time $D_t := \inf\{t \le u \le T; Y_u = L_u\} \wedge T = \inf\{t \le u \le T; N_u = \mathrm{e}^{2L_u}\} \wedge T$, by the continuity of $\overline{K}$, we get $\overline{K}_{D_t} - \overline{K}_t = 0$, which implies $K_{D_t} - K_t = 0$. It follows that

$$\int_0^T (Y_t - L_t)\mathrm{d}K_t = 0.$$

Now we have to prove that $Y_t \in \mathbf{S}^2(0,T)$, $Z_t \in \mathbf{H}_d^2(0,T)$ and $K_t \in \mathbf{A}^2(0,T)$. Using Jensen's inequality we have

$$\begin{aligned} Y_t &= \frac{1}{2}\log N_t = \frac{1}{2}\log[\operatorname{ess}\sup_{\tau\in\mathcal{T}_{t,T}} E[\mathrm{e}^{2\widetilde{L}_\tau}|\mathcal{F}_t]] \\ &\ge \frac{1}{2}\log[\exp(\operatorname{ess}\sup_{\tau\in\mathcal{T}_{t,T}} E[2\widetilde{L}_\tau|\mathcal{F}_t])] \\ &= \operatorname{ess}\sup_{\tau\in\mathcal{T}_{t,T}} E[\widetilde{L}_\tau|\mathcal{F}_t] \ge E[\xi|\mathcal{F}_t] \ge U_t, \end{aligned}$$

with $U_t = -E[\xi^-|\mathcal{F}_t]$. Then for all $a > 0$, define

$$\tau_a = \inf\left\{t;\, |N_t| > a, \int_0^t \left(\frac{\overline{Z}_s}{N_s}\right)^2 \mathrm{d}s > a, \left|\int_0^t \frac{\overline{Z}_s}{N_s}\mathrm{d}B_s\right| > a\right\}.$$

From (8), we get for $0 \le t \le T$

$$\begin{aligned} 0 \le \int_0^t Z_s^2\mathrm{d}s &= Y_0 - Y_t + \int_0^t Z_s\mathrm{d}B_s - K_t \\ &\le Y_0 - U_t + \int_0^t Z_s\mathrm{d}B_s. \end{aligned}$$

Then

$$\left(\int_0^{\tau_a} Z_s^2\mathrm{d}s\right)^2 \le 3(Y_0)^2 + 3(U_{\tau_a})^2 + 3\left(\int_0^{\tau_a} Z_s\mathrm{d}B_s\right)^2.$$

Taking the expectation, using Jensen's inequality and $3x \le \frac{x^2}{2} + \frac{9}{2}$, we obtain

$$\begin{aligned} E\left(\int_0^{\tau_a} Z_s^2\mathrm{d}s\right)^2 &\le \frac{3}{4}(\log N_0)^2 + 3E(\xi^-)^2 + \frac{1}{2}\left(E\left(\int_0^{\tau_a} Z_s^2\mathrm{d}s\right)\right)^2 + \frac{9}{2} \\ &\le \frac{3}{4}(\log N_0)^2 + 3E(\xi^-)^2 + \frac{1}{2}E\left(\int_0^{\tau_a} Z_s^2\mathrm{d}s\right)^2 + \frac{9}{2}, \end{aligned}$$

so

$$E\left(\int_0^{\tau_a} Z_s^2\mathrm{d}s\right)^2 \le \frac{3}{2}(\log N_0)^2 + 6E(\xi^-)^2 + 9 \le C.$$

Since $\tau_a \nearrow T$ when $a \to +\infty$, we get to the limit, and with the Schwarz inequality

$$E\int_0^T Z_s^2\mathrm{d}s \le \left(E\left(\int_0^T Z_s^2\mathrm{d}s\right)^2\right)^{\frac{1}{2}} \le C.$$

So $Z \in \mathbf{H}_d^2(0,T)$. Now from (8), we get for $0 \le t \le T$

$$0 \le K_t = Y_0 - Y_t + \int_0^t Z_s \mathrm{d}B_s - \int_0^t Z_s^2 \mathrm{d}s \le Y_0 - Y_t + \int_0^t Z_s \mathrm{d}B_s.$$

Notice that K is increasing, so it is sufficient to prove $E[K_T^2] < +\infty$. Squaring the inequality on both sides and taking the expectation, we obtain

$$E[(K_T)^2] \le 3Y_0^2 + 3E[\xi^2] + 3E\int_0^T Z_s^2 \mathrm{d}s \le C.$$

Finally, still from (8),

$$Y_t = Y_0 - K_t + \int_0^t Z_s \mathrm{d}B_s - \int_0^t Z_s^2 \mathrm{d}s,$$

so

$$(Y_t)^2 \le 4\,(Y_0)^2 + 4\,(K_t)^2 + 4\left(\int_0^t Z_s \mathrm{d}B_s\right)^2 + 4\left(\int_0^t Z_s^2 \mathrm{d}s\right)^2.$$

Then by the Burkholder–Davis–Gundy inequality, we get

$$\begin{aligned} E[\sup_{0\le t\le T}(Y_t)^2] &\le 4\,(Y_0)^2 + 4E[K_T^2] + 4E\left[\sup_{0\le t\le T}\left(\int_0^t Z_s \mathrm{d}B_s\right)^2\right] + 4E\left(\int_0^T Z_s^2 \mathrm{d}s\right)^2 \\ &\le 4\,(Y_0)^2 + 4E[K_T^2] + CE\left(\int_0^t Z_s^2 \mathrm{d}B_s\right) + 4E\left(\int_0^T Z_s^2 \mathrm{d}s\right)^2 \le C, \end{aligned}$$

i.e. $Y \in \mathbf{S}^2(0,T)$. □

5. The case f linear increasing in z

In this section, we assume that f satisfies

Assumption 6. (i) $f(\cdot, y, z)$ is progressively measurable, and $E\int_0^T f^2(t,0,0)\mathrm{d}t$ is finite;
(ii) there exists $\mu \in \mathbb{R}$, such that $\forall (t,z) \in [0,T] \times \mathbb{R}^d$ and $y, y' \in \mathbb{R}$,

$$(y - y')(f(t,y,z) - f(t,y',z)) \le \mu(y-y')^2;$$

(iii) there exists a nonnegative, continuous, increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, with $\varphi(0) = 0$, s.t. $\forall (t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$,

$$|f(t,y,z)| \le |g_t| + \varphi(|y|) + \beta|z|,$$

where $g_t \in \mathbf{H}^2(0,T)$;
(iv) for all $t \in [0,T]$, $(y,z) \to f(t,y,z)$ is continuous.

For $\varphi(x) = |x|$, i.e. f linear increasing in y and z, Matoussi proved in [9] that when $\xi \in \mathbf{L}^2(\mathcal{F}_T)$ and $L \in \mathbf{S}^2(0,T)$, there exists a triple (Y, Z, K) which is solution of the RBSDE(ξ, f, L).

The result of this section is the following:

Theorem 5.1. *Suppose that $\xi \in \mathbf{L}^2(\mathcal{F}_T)$, f and L satisfy respectively Assumptions 6 and 3; then the RBSDE(ξ, f, L) has a minimal solution.*

First we note that (Y, Z, K) solves the RBSDE(ξ, f, L) if and only if

$$(\overline{Y}_t, \overline{Z}_t, \overline{K}_t) := \left(e^{\lambda t} Y_t, e^{\lambda t} Z_t, \int_0^t e^{\lambda s} dK_s\right) \tag{9}$$

solves the RBSDE$(\overline{\xi}, \overline{f}, \overline{L})$, where

$$(\overline{\xi}, \overline{f}(t, y, z), \overline{L}_t) = (\xi e^{\lambda T}, e^{\lambda t} f(t, e^{-\lambda t} y, e^{-\lambda t} z) - \lambda y, e^{\lambda t} L_t).$$

If we choose $\lambda = \mu$, then the coefficient $\overline{f}$ satisfies the same assumptions as in Assumption 6, with (ii) replaced by

(ii$'$) $(y - y')(f(t, y, z) - f(t, y', z)) \leq 0$.

Since we are in the one-dimensional case, (ii$'$) means that f is decreasing in y. From another part, $\overline{\xi}$ still belongs to $\mathbf{L}^2(\mathcal{F}_T)$ and the barrier $\overline{L}$ still satisfies Assumption 3. So in the following, we shall work under Assumption 6$'$ with (ii) replaced by (ii$'$).

We need first an estimation result and a monotonic stability theorem.

Lemma 5.1. *We consider RBSDE(ξ, g, L), with $\xi \in \mathbf{L}^2(\mathcal{F}_T)$; g and L satisfy Assumptions 6$'$ and 3. Moreover $g(t, y, z)$ is Lipschitz in z. Then we have the following estimation:*

$$E\left[\sup_{0 \leq t \leq T} |y_t|^2 + \int_0^T |z_s| ds + |k_T|^2\right] \leq C_\beta E\left[|\xi|^2 + \int_0^T g_s^2 ds + \varphi^2(b) + \varphi^2(2T) + 1\right].$$

Here $(y_t, z_t, k_t)_{0 \leq t \leq T}$ is solution of RBSDE(ξ, g, L), C_β is a constant which only depends on β, T and b.

Remark 5.1. The constant C_β does not depend on the Lipschitz coefficient of g on z.

Sketch of proof. Since g is Lipschitz in z, by the Theorem 2 in [7], the RBSDE(ξ, g, L) admits the unique solution $(y_t, z_t, k_t)_{0 \leq t \leq T}$. Applying Itô's formula to $|y_t|^2$, with classical techniques and Gronwall's inequality, we know that there exists a constant c_1 depending on β and T such that for $t \in [0, T]$,

$$E[|y_t|^2] \leq c_1 E\left[|\xi|^2 + \int_0^T g_s^2 ds + b(k_T - k_t)\right] \tag{10}$$

and

$$E\left[\int_t^T |z_s|^2 ds\right] \leq 2(1 + (1 + 2\beta^2)T) c_1 E\left[|\xi|^2 + \int_0^T g_s^2 ds + b(k_T - k_t)\right]. \tag{11}$$

Then we need to estimate the increasing process k. By the same approximation methods as were used in the proof of Theorem 2 in [7], we can prove that there exists a constant only depending on β, b and T s.t.

$$E[(k_T - k_t)^2] \leq 2c_6 E\left[|\xi|^2 + 2\int_t^T g_s^2 ds + \varphi^2(b) + \varphi^2(2T) + 2c_6 b^2 + 1\right].$$

So the results follow. $\square$

Proof of Theorem 5.1. The proof consists of four steps.

Step 1. Approximation. For $n \geq \beta$, we introduce the following functions:

$$f_n(t, y, z) = \inf_{q \in \mathbf{Q}^d} \{f(t, y, q) + n|z - q|\}.$$

Then we have

1. for all (t, z), $y \to f_n(t, y, z)$ is non-increasing;
2. for all (t, y), $z \to f_n(t, y, z)$ is n-Lipschitz;
3. for all (t, y, z), $|f_n(t, y, z)| \leq |g_t| + \varphi(|y|) + \beta\,|z|$.

Thanks to the results of [7], we know that for each $n \geq \beta$, there exists a unique triple (Y^n, Z^n, K^n) which is solution of the RBSDE(ξ, f_n, L).

Step 2. Estimation results. Let $\alpha \geq 0$ be a real number which will be chosen later. We set

$$U_t^n = \mathrm{e}^{\alpha t} Y_t^n, \qquad V_t^n = \mathrm{e}^{\alpha t} Z_t^n, \qquad \mathrm{d}J_t^n = \mathrm{e}^{\alpha t}\mathrm{d}K_t^n.$$

Then we know that (U^n, V^n, J^n) is the solution of the RBSDE(ζ, F_n, L^α), where

$$\zeta = \mathrm{e}^{\alpha T}\xi,\ F_n(t, u, v) = \mathrm{e}^{\alpha t} f_n(t, \mathrm{e}^{-\alpha t}u, \mathrm{e}^{-\alpha t}v) - \alpha u, \qquad L_t^\alpha = \mathrm{e}^{\alpha t} L_t.$$

It is easy to check that

$$|F_n(t, u, v)| \leq \mathrm{e}^{\alpha t}|g_t| + \mathrm{e}^{\alpha t}\varphi(|u|) + \alpha|u| + \beta|v|,$$

Setting $\psi(u) = \mathrm{e}^{\alpha T}\varphi(|u|) + \alpha|u|$, with $\psi(u) = 0$, we get that F_n satisfies

Assumption 6′. (iii). Moreover

$$\begin{aligned} uF_n(t, u, v) &= \mathrm{e}^{\alpha t} u f_n(t, \mathrm{e}^{-\alpha t}u, \mathrm{e}^{-\alpha t}v) - \alpha u^2 \\ &\leq u\mathrm{e}^{\alpha t} g_t + \beta\,|u|\,|v| - \alpha u^2, \end{aligned}$$

and $\sup_{0\leq t\leq T} L_t^\alpha \leq \mathrm{e}^{\alpha T}\sup_{0\leq t\leq T} L_t \leq \mathrm{e}^{\alpha T} b$. If we apply the Itô formula to $|U^n|^2$ on $[t, T]$, and take the conditional expectation, then we get

$$\begin{aligned} \left|U_t^n\right|^2 + \frac{1}{2}E\left[\int_t^T \left|V_s^n\right|^2 \mathrm{d}s|\mathcal{F}_t\right] &\leq E\left[|\zeta|^2 + \int_t^T \mathrm{e}^{2\alpha s} g_s^2 \mathrm{d}s + \theta\mathrm{e}^{2\alpha T} b^2|\mathcal{F}_t\right] \\ &+ (1 + 2\beta^2 - \alpha)E\left[\int_t^T \left|U_s^n\right|^2 \mathrm{d}s|\mathcal{F}_t\right] + \frac{1}{\theta}E[(J_T^n - J_t^n)^2|\mathcal{F}_t] \end{aligned} \tag{12}$$

where θ is a constant which will be decided on later. Since

$$J_T^n - J_t^n = U_t^n - \zeta - \int_t^T F_n(s, U_s^n, V_s^n)\mathrm{d}s - \int_t^T V_s^n \mathrm{d}B_s,$$

we have

$$E[(J_T^n - J_t^n)^2|\mathcal{F}_t] \leq 4|U_t^n|^2 + 4E\left[|\zeta|^2 + \left(\int_t^T F_n(s, U_s^n, V_s^n)\mathrm{d}s\right)^2 + \int_t^T |V_s^n|^2\mathrm{d}s|\mathcal{F}_t\right].$$

Using the same approximation as in Theorem 2 in [7] or Lemma 5.1, except considering conditional expectation $E[\cdot|\mathcal{F}_t]$ instead of the expectation, we deduce

$$E[(J_T^n - J_t^n)^2|\mathcal{F}_t] \leq c_\beta E\left[|\zeta|^2 + \int_t^T \mathrm{e}^{2\alpha s} g_s^2 \mathrm{d}s + \psi^2(\mathrm{e}^{\alpha T} b) + \psi^2(2T) + 1|\mathcal{F}_t\right],$$

where c_β is a constant which only depends on β, T, b and α. If we substitute it into (12), and set $\alpha = 1 + 2\beta^2$, $\theta = c_\beta$, then we get

$$\begin{aligned} \left|U_t^n\right|^2 &\leq 2E\left[|\zeta|^2 + \int_t^T F_n^2(s, 0, 0)\mathrm{d}s|\mathcal{F}_t\right] + \mathrm{e}^{\alpha T}(\varphi(\mathrm{e}^{\alpha T} b) + \varphi(2T)) \\ &\quad + \alpha(\mathrm{e}^{\alpha T} b + 2T) + 1 + c_\beta \mathrm{e}^{2\alpha T} b^2. \end{aligned}$$

From the definition of U^n, we get

$$\left|Y_t^n\right|^2 \le \mathrm{e}^{-2\alpha t}\left(2E\left[\mathrm{e}^{2\alpha T}|\xi|^2+\int_t^T \mathrm{e}^{2\alpha s}g_s^2\mathrm{d}s|\mathcal{F}_t\right]+\mathrm{e}^{\alpha T}(\varphi(\mathrm{e}^{\alpha T}b)+\varphi(2T))\right.$$
$$\left.+\alpha(\mathrm{e}^{\alpha T}b+2T)+1+c_\beta \mathrm{e}^{2\alpha T}b^2\right).$$

Finally we set $M_t = (\mathrm{e}^{2\alpha T}2E[|\xi|^2+\int_t^T g_s^2\mathrm{d}s|\mathcal{F}_t]+\mathrm{e}^{\alpha T}(\varphi(\mathrm{e}^{\alpha T}b)+\varphi(2T))+c_\beta\mathrm{e}^{2\alpha T}b^2+\alpha(\mathrm{e}^{\alpha T}b+2T)+1)^{\frac{1}{2}}$; then we get

$$\left|Y_t^n\right| \le M_t, \quad \forall t\in[0,T]. \tag{13}$$

Step 3. Localization.

Since the sequence $(f_n)_{n\ge\beta}$ is non-decreasing in n, then from the comparison theorem in [7], we get

$$Y_t^n \le Y_t^{n+1}, \quad \forall t\in[0,T],\ \forall n\ge\beta.$$

Define $Y_t = \sup_{n\ge\beta} Y_t^n$.

We now consider a localization procedure. For $m\in\mathbb{N}, m\ge b$, let τ_m be the following stopping time:

$$\tau_m = \inf\{t\in[0,T] : M_t+g_t\ge m\}\wedge T,$$

and we introduce the stopped process $Y_t^{n,m} = Y_{t\wedge\tau_m}^n$, together with $Z_t^{n,m} = Z_t^n 1_{\{t\le\tau_m\}}$ and $K_t^{n,m} = K_{t\wedge\tau_m}^n$. Then $(Y_t^{n,m}, Z_t^{n,m}, K_t^{n,m})_{0\le t\le T}$ solves the following RBSDE:

$$Y_t^{n,m} = \xi^{n,m} + \int_t^T 1_{\{s\le\tau_m\}} f_n(s, Y_s^{n,m}, Z_s^{n,m})\mathrm{d}s + K_T^{n,m} - K_t^{n,m} - \int_t^T Z_s^{n,m}\mathrm{d}B_s,$$
$$Y_t^{n,m}\ge L_t, \quad \int_0^T (Y_t^{n,m}-L_t)\mathrm{d}K_t^{n,m} = 0.$$

Here $\xi^{n,m} = Y_{\tau_m}^{n,m} = Y_{\tau_m}^n$.

Since $(Y^{n,m})_{n\ge\beta}$ is non-decreasing in n, with (13), we get $\sup_{n\ge\beta}\sup_{t\in[0,T]}\left|Y_t^{n,m}\right|\le m$. If we set $\rho_m(y) = \frac{ym}{\max\{|y|,m\}}$, it is easy to check that $(Y^{n,m}, Z^{n,m}, K^{n,m})$ satisfies

$$Y_t^{n,m} = \xi^{n,m} + \int_t^T 1_{\{s\le\tau_m\}} f_n(s, \rho_m(Y_s^{n,m}), Z_s^{n,m})\mathrm{d}s + K_T^{n,m} - K_t^{n,m} - \int_t^T Z_s^{n,m}\mathrm{d}B_s,$$
$$Y_t^{n,m}\ge L_t, \quad \int_0^T (Y_t^{n,m}-L_t)\mathrm{d}K_t^{n,m} = 0.$$

Moreover, we have

$$\left|1_{\{s\le\tau_m\}} f_n(s,\rho_m(y),z)\right| \le m+\varphi(m)+\beta|z|,$$

and $|\xi^{n,m}|\le m$. From Dini's theorem, we know that $1_{\{s\le\tau_m\}} f_n(s,\rho_m(y),z)$ converges increasingly to $1_{\{s\le\tau_m\}} f(s,\rho_m(y),z)$ uniformly on the compact sets of $\mathbb{R}\times\mathbb{R}^d$, since f_n are continuous and converge increasingly to f. Also $\xi^{n,m}$ converge increasingly to ξ^m a.s., where $\xi^m = \sup_{n\ge\beta}\xi^{n,m}$.

As in [9], we can prove that $Y^{n,m}$ converges increasingly to Y^m in $\mathbf{S}^2(0,T)$, and $Z^{n,m}\to Z^m$ in $\mathbf{H}_d^2(0,T)$, $K^{n,m}\searrow K^m$ uniformly on $[0,T]$. Moreover, (Y^m, Z^m, K^m) solves the following

RBSDE:

$$Y_t^m = \xi^m + \int_t^T 1_{\{s\le\tau_m\}} f(s, \rho_m(Y_s^m), Z_s^m)\mathrm{d}s + K_T^m - K_t^m - \int_t^T Z_s^m \mathrm{d}B_s,$$
$$Y_t^m \ge L_t, \quad \int_0^T (Y_t^m - L_t)\mathrm{d}K_t^m = 0,$$

where $\xi^m = \sup_{n\ge\beta} Y_{\tau_m}^{n,m}$. Notice that $\left|Y_t^m\right| \le m$, so we have

$$Y_t^m = \xi^m + \int_t^T 1_{\{s\le\tau_m\}} f(s, Y_s^m, Z_s^m)\mathrm{d}s + K_T^m - K_t^m - \int_t^T Z_s^m \mathrm{d}B_s.$$

From the definition of $\{\tau_m\}$, it is easy to check that $\tau_m \le \tau_{m+1}$; with the definition of Y^m, Z^m, K^m and Y, we get

$$Y_{t\wedge\tau_m} = Y_{t\wedge\tau_m}^{m+1} = Y_t^m, \qquad Z_t^{m+1} 1_{\{t\le\tau_m\}} = Z_t^m, \qquad K_{t\wedge\tau_m}^{m+1} = K_t^m.$$

If we define

$$Z_t := Z_t^1 1_{\{t\le\tau_1\}} + \sum_{m\ge 2} Z_t^m 1_{(\tau_{m-1},\tau_m]}(t), \qquad K_{t\wedge\tau_m} := K_t^m,$$

since the processes (Y^m) are continuous, and P-a.s. $\tau_m = T$, for m large enough, then Y is continuous on $[0, T]$. It follows that K is also continuous on $[0, T]$. Furthermore, we have for $m \in \mathbb{N}$,

$$Y_{t\wedge\tau_m} = Y_{\tau_m} + \int_{t\wedge\tau_m}^{\tau_m} f(s, Y_s, Z_s)\mathrm{d}s + K_{\tau_m} - K_{t\wedge\tau_m} - \int_{t\wedge\tau_m}^{\tau_m} Z_s \mathrm{d}B_s. \tag{14}$$

Finally, we have

$$\begin{aligned} P\left(\int_0^T |Z_s|^2\,\mathrm{d}s = \infty\right) &= P\left(\int_0^T |Z_s|^2\,\mathrm{d}s = \infty, \tau_m = T\right) \\ &\quad + P\left(\int_0^T |Z_s|^2\,\mathrm{d}s = \infty, \tau_m < T\right) \\ &\le P\left(\int_0^T |Z_s|^2\,\mathrm{d}s = \infty\right) + P(\tau_m < T), \end{aligned}$$

and in the same way,

$$P(|K_T|^2 = \infty) \le P(|K_T|^2 = \infty) + P(\tau_m < T).$$

Since $\tau_m \nearrow T$, P-a.s., we know that $\int_0^T |Z_s|^2\,\mathrm{d}s < \infty$ and $|K_T|^2 < \infty$, P-a.s. Letting $m \to \infty$ in (14), we get that (Y, Z, K) satisfies the equation RBSDE(ξ, f, L).

Step 4. We want to prove that the triple (Y, Z, K) is a solution of the RBSDE(ξ, f, L).

We consider the integrability of (Y, Z, K). By (13), we know that for $0 \le t \le T$, $|Y_t| \le M_t$. It follows that

$$E[\sup_{0\le t\le T} |Y_t|^2] \le C_\beta E\left[|\xi|^2 + \int_0^T g_s^2\mathrm{d}s + \varphi^2(b) + \varphi^2(2T) + 1\right],$$

where C_β is a constant which depends only on β, T and b. For K, notice that $K^{n,m} \searrow K^m$; then for each $m \in \mathbb{N}$, $0 \le t \le T$, we know that $0 \le K_t^m \le K_t^{1,m}$. Obviously, the coefficient

$1_{\{s\leq\tau_m\}} f_n(s,\rho_m(y),z)$ satisfies Assumption 6′, and is Lipschitz in z; then by Lemma 5.1, with (13), we have

$$E[(K_T^{1,m})^2] \leq 2C_\beta E\left[|\xi|^2 + \int_0^T g_s^2 \mathrm{d}s + \varphi^2(b) + \varphi^2(2T) + 1\right].$$

It follows that for each $m \in \mathbb{N}$,

$$E[(K_T^m)^2] \leq 2C_\beta E\left[|\xi|^2 + \int_0^T g_s^2 \mathrm{d}s + \varphi^2(b) + \varphi^2(2T) + 1\right],$$

which implies the same for K, i.e. we get $E[(K_T)^2] < \infty$.

In order to estimate Z, we apply Itô's formula to $|Y_t|^2$ on the interval $[0,T]$; with the estimates on Y and K, there exists a constant C which only depends on β, T and b such that

$$E\int_0^T |Z_s|^2\,\mathrm{d}s \leq CE\left[|\xi|^2 + \int_0^T g_s^2 \mathrm{d}s + \varphi^2(b) + \varphi^2(2T) + 1\right].$$

The last thing to check is the integral condition. From the fact that $\int_0^T (Y_t^m - L_t)\mathrm{d}K_t^m = 0$, and that P-a.s. $\tau_m = T$, for m large enough, we get

$$\int_0^T (Y_t - L_t)\mathrm{d}K_t = 0, \text{ a.s.},$$

i.e. (Y,Z,K) is a solution of RBSDE(ξ,f,L) in $\mathbf{S}^2(0,T)\times\mathbf{H}_d^2(0,T)\times\mathbf{A}^2(0,T)$. □

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